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ON THE SEMIGROUP OF ALL MONOID ENDOMORPHISMS OF THE SEMIGROUP $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ WITH THE TWO-ELEMENTS FAMILY $\mathscr{F}$ OF INDUCTIVE NONEMPTY SUBSETS OF ω

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We study the structure of the semigroup $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of all monoid endomorphisms of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, where an ω-closed family $\mathscr{F}$ consists of two nonempty inductive subsets of ω. We describe elements of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$, the multiplication and Green's relations on the semigroup $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$.

Key words: Bicyclic monoid, inverse semigroup, bicyclic extension, endomorphism, ideal, Green's relations.

1. Introduction, motivation and main definitions

We shall follow the terminology of [1, 2, 11]. By ω we denote the set of all nonnegative integers, by $\mathbb{N}$ the set of all positive integers.

Let $\mathscr{P}(\omega)$ be the family of all subsets of ω. For any $F \in \mathscr{P}(\omega)$ and $n \in \mathbb{Z}$ we put $n + F = \{n + k : k \in F\}$ if $F \neq \varnothing$ and $n + \varnothing = \varnothing$. A subfamily $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is called *ω-closed* if $F_1 \cap (-n + F_2) \in \mathscr{F}$ for all $n \in \omega$ and $F_1, F_2 \in \mathscr{F}$. For any $a \in \omega$ we denote $[a) = \{x \in \omega : x \geqslant a\}$.

A subset A of ω is said to be *inductive*, if $i \in A$ implies $i + 1 \in A$. Obviously, that $\varnothing$ is an inductive subset of ω.

Зауваження 1 ([5]). (1) By Lemma 6 from [4] nonempty subset $F \subseteq \omega$ is inductive in ω if and only if $(-1 + F) \cap F = F$.

(2) Since the set ω with the usual order is well-ordered, for any nonempty inductive subset F in ω there exists nonnegative integer $n_F \in \omega$ such that $[n_F) = F$.

(3) Statement (2) implies that the intersection of an arbitrary finite family of nonempty inductive subsets in ω is a nonempty inductive subset of ω.

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Let S be an arbitrary semigroup. By S^1 we denote S with the enjoined unit element 1. Any homomorphism $\alpha\colon S\to S$ is called an *endomorphism* of S [1]. If the semigroup has the identity element 1_S then the endomorphism α of S such that $(1_S)\alpha=1_S$ is said to be a *monoid endomorphism* of S. A bijective endomorphism of S is called an *automorphism*. Traditionary in the semigroup theory the image of an element s of a semigroup S under the endomorphism $\alpha\colon S\to S$ is denoted by $(s)\alpha$.

A semigroup S is called *inverse* if for any element $x\in S$ there exists a unique $x^{-1}\in S$ such that $xx^{-1}x=x$ and $x^{-1}xx^{-1}=x^{-1}$. The element x^{-1} is called the *inverse of* $x\in S$. If S is an inverse semigroup, then the function $\operatorname{inv}\colon S\to S$ which assigns to every element x of S its inverse element x^{-1} is called the *inversion*.

If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a *band* (or the *band of* S). Then the semigroup operation on S determines the following partial order $\preccurlyeq$ on $E(S)$: $e\preccurlyeq f$ if and only if $ef=fe=e$. This order is called the *natural partial order* on $E(S)$.

If S is an inverse semigroup then the semigroup operation on S determines the following partial order $\preccurlyeq$ on S: $s\preccurlyeq t$ if and only if there exists $e\in E(S)$ such that $s=te$. This order is called the *natural partial order* on S [15].

If S is a semigroup, then we shall denote the Green relations on S by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$ and $\mathscr{H}$ (see [1, Section 2.1]):

$$
\begin{aligned}
&a\mathscr{R}b \text{ if and only if } aS^1=bS^1;\\
&a\mathscr{L}b \text{ if and only if } S^1a=S^1b;\\
&a\mathscr{J}b \text{ if and only if } S^1aS^1=S^1bS^1;\\
&\mathscr{D}=\mathscr{L}\circ\mathscr{R}=\mathscr{R}\circ\mathscr{L};\\
&\mathscr{H}=\mathscr{L}\cap\mathscr{R}.
\end{aligned}
$$

The $\mathscr{L}$-class [$\mathscr{R}$-class, $\mathscr{H}$-class, $\mathscr{D}$-class, $\mathscr{J}$-class] of the semigroup S containing the element $a\in S$ will be denoted by $\boldsymbol{L}_a$ [$\boldsymbol{R}_a$, $\boldsymbol{H}_a$, $\boldsymbol{D}_a$, $\boldsymbol{J}_a$].

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq=1$. The semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$
q^kp^l\cdot q^mp^n=q^{k+m-\min\{l,m\}}p^{l+n-\min\{l,m\}}.
$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence [1].

On the set $\boldsymbol{B}_\omega=\omega\times\omega$ we define the semigroup operation "$\cdot$" in the following way

$$
(i_1,j_1)\cdot(i_2,j_2)=\begin{cases}(i_1-j_1+i_2,j_2), & \text{if } j_1\leqslant i_2;\\ (i_1,j_1-i_2+j_2), & \text{if } j_1\geqslant i_2.\end{cases} \tag{1}
$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ to the semigroup $\boldsymbol{B}_\omega$ is isomorphic by the mapping $\mathfrak{h}\colon\mathscr{C}(p,q)\to\boldsymbol{B}_\omega$, $q^kp^l\mapsto(k,l)$ (see: [1, Section 1.12] or [13, Exercise IV.1.11(ii)]).

Next we shall describe the construction which is introduced in [4].

Let $\mathscr{F}$ be an ω-closed subfamily of $\mathscr{P}(\omega)$. On the set $\boldsymbol{B}_{\omega}\times\mathscr{F}$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1, j_1, F_1)\cdot(i_2, j_2, F_2) = \begin{cases} (i_1 - j_1 + i_2, j_2, (j_1 - i_2 + F_1)\cap F_2), & \text{if } j_1 \leqslant i_2;\\ (i_1, j_1 - i_2 + j_2, F_1 \cap (i_2 - j_1 + F_2)), & \text{if } j_1 \geqslant i_2. \end{cases} \qquad (2)$$

In [4] is proved that $(\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot)$ is a semigroup. Moreover, if $\mathscr{F}$ contains the empty set $\varnothing$ then the set $\boldsymbol{I} = \{(i,j,\varnothing)\colon i,j\in\omega\}$ is an ideal of the semigroup $(\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot)$. For any ω-closed family $\mathscr{F}\subseteq\mathscr{P}(\omega)$ the following semigroup

$$\boldsymbol{B}_{\omega}^{\mathscr{F}} = \begin{cases} (\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot)/\boldsymbol{I}, & \text{if } \varnothing\in\mathscr{F};\\ (\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot), & \text{if } \varnothing\notin\mathscr{F} \end{cases}$$

is defined in [4]. The semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ generalizes the bicyclic monoid and the countable semigroup of matrix units. It is proven in [4] that $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a combinatorial inverse semigroup and Green's relations, the natural partial order on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and its set of idempotents are described. In [4] the criteria when the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is simple, 0-simple, bisimple, 0-bisimple, or it has the identity, are given. In particularly in [4] it is proved that the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the semigrpoup of $\omega\times\omega$-matrix units if and only if $\mathscr{F}$ consists of a singleton set and the empty set, and $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the bicyclic monoid if and only if $\mathscr{F}$ consists of a non-empty inductive subset of ω.

Group congruences on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and its homomorphic retracts in the case when an ω-closed family $\mathscr{F}$ consists of inductive non-empty subsets of ω are studied in [5]. It is proven that a congruence $\mathfrak{C}$ on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a group congruence if and only if its restriction on a subsemigroup of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, which is isomorphic to the bicyclic semigroup, is not the identity relation. Also in [5], all non-trivial homomorphic retracts and isomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ are described. In [6] it is proved that an injective endomorphism ε of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is the indentity transformation if and only if ε has three distinct fixed points, which is equivalent to existence non-idempotent element $(i,j,[p))\in\boldsymbol{B}_{\omega}^{\mathscr{F}}$ such that $(i,j,[p))\varepsilon = (i,j,[p))$.

In [3, 12] the algebraic structure of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is established in the case when ω-closed family $\mathscr{F}$ consists of atomic subsets of ω.

It is well-known that every automorphism of the bicyclic monoid $\boldsymbol{B}_{\omega}$ is the identity self-map of $\boldsymbol{B}_{\omega}$ [1], and hence the group $\mathbf{Aut}(\boldsymbol{B}_{\omega})$ of automorphisms of $\boldsymbol{B}_{\omega}$ is trivial. In [10] it is proved that the semigroups $\mathbf{End}(\boldsymbol{B}_{\omega})$ of the endomorphisms of the bicyclic semigroup $\boldsymbol{B}_{\omega}$ is isomorphic to the semidirect products $(\omega,+)\rtimes_{\varphi}(\omega,*)$, where $+$ and $*$ are the usual addition and the usual multiplication on the set of non-negative integers ω.

In the paper [7] the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is studied, where the family $\mathscr{F}_n$ is generated by initial interval $[0,n]$ of ω. In the paper [14] the semigroup $\boldsymbol{End}(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$ of all endomorphisms of the monoid $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ is described up to its ideal $\boldsymbol{End}^1(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$, where $\boldsymbol{End}^1(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$ is the subsemigroup of $\boldsymbol{End}(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$ which consists of $\mathfrak{a}\in\boldsymbol{End}(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$ such that the image $(\alpha)\mathfrak{a}$ is isomorphic to a subsemigroup of the semigroup of $\omega\times\omega$-matrix units for all $\alpha\in\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$.

In the paper [8] we study injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with the two-elements family $\mathscr{F}$ of inductive nonempty subsets of ω. We describe elements of the semigroup $\boldsymbol{End}_*^1(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of all injective monoid endomorphisms of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. Also in [8] we prove that Green's relations $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, $\mathscr{D}$, and $\mathscr{J}$ on $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ coincide with the relation of equality.

In the paper [9] we study non-injective endomorphisms of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$. We describe the structure of elements of the semigroup $\boldsymbol{End}_0^*(\boldsymbol{B}_\omega^{\mathscr{F}})$ of monoid non-injective endomorphisms of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$. In particular we show that its subsemigroup $\boldsymbol{End}^*(\boldsymbol{B}_\omega^{\mathscr{F}})$ of monoid non-injective non-annihilating endomorphisms of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ is isomorphic to the direct product the two-element left-zero semigroup and the multiplicative semigroup of positive integers and describe Green's relations on $\boldsymbol{End}^*(\boldsymbol{B}_\omega^{\mathscr{F}})$.

This paper is a continuation of [8] and [9]. We study the structure of the semigroup $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$ of all monoid endomorphisms of $\boldsymbol{B}_\omega^{\mathscr{F}}$. We describe elements of $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$, the multiplication and Green's relations on the semigroup $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$.

Later we assume that an ω-closed family $\mathscr{F}$ consists of two nonempty inductive nonempty subsets of ω.

2. Preliminary results on endomorphisms of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$

Fix an arbitrary positive integer k and any $p \in \{0, \ldots, k-1\}$. For all $i, j \in \omega$ we define the transformation $\alpha_{k,p}$ of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ in the following way

$$(i, j, [0])\alpha_{k,p} = (ki, kj, [0]),$$
$$(i, j, [1])\alpha_{k,p} = (p + ki, p + kj, [1]).$$

It is obvious that $\alpha_{k,p}$ is an injective transformation of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$ (see [8, Example 1 and Lemma 2]).

Fix an arbitrary positive integer $k \geqslant 2$ and any $p \in \{1, \ldots, k-1\}$. For all $i, j \in \omega$ we define the transformation $\beta_{k,p}$ of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ in the following way

$$(i, j, [0])\beta_{k,p} = (ki, kj, [0]),$$
$$(i, j, [1])\beta_{k,p} = (p + ki, p + kj, [0]).$$

It is obvious that $\beta_{k,p}$ is an injective transformation of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$ (see [8, Example 2 and Lemma 3]).

The following theorem from [8] describes the structure of the semigroup $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ of all injective monoid endomorphisms of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ for the family $\mathscr{F} = \{[0), [1)\}$.

Theorem 1 ([8, Theorem 1]). *Let $\mathscr{F} = \{[0), [1)\}$ and ε be an injective monoid endomorphism of $\boldsymbol{B}_\omega^{\mathscr{F}}$. Then either there exist a positive integer k and $p \in \{0, \ldots, k-1\}$ such that $\varepsilon = \alpha_{k,p}$ or there exist a positive integer $k \geqslant 2$ and $p \in \{1, \ldots, k-1\}$ such that $\varepsilon = \beta_{k,p}$. Moreover, the following statements hold:*

(1) *$(i, j, [0))\alpha_{k,p} = (i, j, [0))\beta_{k,p}$ for all $i, j \in \omega$ and any positive integer $k \geqslant 2$ and $p \in \{1, \ldots, k-1\}$;*
(2) *if $k = 1$ then $\varepsilon = \alpha_{1,0}$ is an automorphism of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$ which is the identity selfmap of $\boldsymbol{B}_\omega^{\mathscr{F}}$;*
(3) *$\alpha_{k_1,p_1}\alpha_{k_2,p_2} = \alpha_{k_1k_2,p_2+k_2p_1}$ for all positive integers k_1, k_2, any $p_1 \in \{0, \ldots, k_1-1\}$, and any $p_2 \in \{0, \ldots, k_2-1\}$;*

(4) $\alpha_{k_1,p_1}\beta_{k_2,p_2} = \beta_{k_1k_2,p_2+k_2p_1}$ *for all positive integers* k_1 *and* $k_2 \geqslant 2$, *any* $p_1 \in \{0,\ldots,k_1-1\}$, *and any* $p_2 \in \{1,\ldots,k_2-1\}$;
(5) $\beta_{k_1,p_1}\beta_{k_2,p_2} = \beta_{k_1k_2,k_2p_1}$ *for all positive integers* $k_1, k_2 \geqslant 2$, *any* $p_1\in\{1,\ldots,k_1-1\}$, *and any* $p_2 \in \{1,\ldots,k_2-1\}$;
(6) $\beta_{k_1,p_1}\alpha_{k_2,p_2} = \beta_{k_1k_2,k_2p_1}$ *for all positive integers* $k_1 \geqslant 2$ *and* k_2, *any* $p_1 \in \{1,\ldots,k_1-1\}$, *and any* $p_2 \in \{0,\ldots,k_2-1\}$;
(7) *if* α_{k_2,p_2}, β_{k_1,p_1}, *and* β_{k_2,p_2} *are well defined elements of* $\boldsymbol{End}_*^1(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ *then* $\beta_{k_1,p_1}\alpha_{k_2,p_2} = \beta_{k_1,p_1}\beta_{k_2,p_2}$;
(8) $\alpha_{1,0}$ *is the unique idempotent of* $\boldsymbol{End}_*^1(\boldsymbol{B}_{\omega}^{\mathscr{F}})$, *and moreover* $\alpha_{1,0}$ *is the identity of* $\boldsymbol{End}_*^1(\boldsymbol{B}_{\omega}^{\mathscr{F}})$;
(9) $S_\alpha = \langle \alpha_{k,p} \mid k\in\mathbb{N}, p\in\{0,\ldots,k-1\}\rangle$ *is a cancellative submonoid of* $\boldsymbol{End}_*^1(\boldsymbol{B}_{\omega}^{\mathscr{F}})$;
(10) $S_\beta = \langle \beta_{k,p} \mid k=2,3,\ldots, p\in\{1,\ldots,k-1\}\rangle$ *is an ideal in* $\boldsymbol{End}_*^1(\boldsymbol{B}_{\omega}^{\mathscr{F}})$.

Let k be an arbitrary non-negative integer. We define maps $\gamma_k\colon \boldsymbol{B}_{\omega}^{\mathscr{F}} \to \boldsymbol{B}_{\omega}^{\mathscr{F}}$ and $\delta_k\colon \boldsymbol{B}_{\omega}^{\mathscr{F}} \to \boldsymbol{B}_{\omega}^{\mathscr{F}}$ by the formulae

$$(i,j,[0])\gamma_k = (i,j,[1])\gamma_k = (ki,kj,[0])$$

and

$$(i,j,[0])\delta_k = (ki,kj,[0]) \qquad \text{and} \qquad (i,j,[1])\delta_k = (k(i+1),k(j+1),[0])$$

for all $i,j\in\omega$, respectively. Then for any $k\in\omega$ the maps γ_k and δ_k are non-injective monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ [9, Examples 1 and 2, Proposition 1].

Remark 1. It obvious that if $\mathfrak{e}$ is the annihilating endomorphism of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ then $\mathfrak{e} = \gamma_0 = \delta_0$.

By $\boldsymbol{End}_0^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ we denote the semigroup of all non-injective monoid endomorphisms of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ for the family $\mathscr{F} = \{[0),[1)\}$.

Theorems 2 and 3 describe the algebraic structure of the semigroup $\boldsymbol{End}_0^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$.

Theorem 2 ([9, Theorem 1])**.** *Let* $\mathscr{F} = \{[0),[1)\}$. *Then for any non-injective monoid endomorphism* $\mathfrak{e}$ *of the monoid* $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ *only one of the following conditions holds:*
(1) $\mathfrak{e}$ *is the annihilating endomorphism, i.e.,* $\mathfrak{e} = \gamma_0 = \delta_0$;
(2) $\mathfrak{e} = \gamma_k$ *for some positive integer* k;
(3) $\mathfrak{e} = \delta_k$ *for some positive integer* k.

Theorem 3 ([9, Theorem 2])**.** *Let* $\mathscr{F} = \{[0),[1)\}$. *Then for all positive integers* k_1 *and* k_2 *the following conditions hold:*
(1) $\gamma_{k_1}\gamma_{k_2} = \gamma_{k_1k_2}$;
(2) $\gamma_{k_1}\delta_{k_2} = \gamma_{k_1k_2}$;
(3) $\delta_{k_1}\gamma_{k_2} = \delta_{k_1k_2}$;
(4) $\delta_{k_1}\delta_{k_2} = \delta_{k_1k_2}$.

By $\mathfrak{e}_{\mathbf{0}}$ we denote the annihilating monoid endomorphism of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ for the family $\mathscr{F} = \{[0),[1)\}$, i.e., $(i,j,[p))\mathfrak{e}_{\mathbf{0}} = (0,0,[0))$ for all $i,j\in\omega$ and $p=0,1$. We put $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}}) = \boldsymbol{End}_0^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})\setminus\{\mathfrak{e}_{\mathbf{0}}\}$. Theorem 3 implies that $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ is a subsemigroup of $\boldsymbol{End}_0^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$.

3. On the semigroup of all monoid endomorphisms of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$

Later by $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ we denote the semigroup of all monoid endomorphisms of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$.

Theorems 1, 3 and 4 describe the semigroup operation on the monoid $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$.

Theorem 4. *Let $\mathscr{F}=\{[0),[1)\}$. Then the following statements hold.*

(1) *If k and n are arbitrary positive integers and $p\in\{0,\ldots,k-1\}$ then*

$$\alpha_{k,p}\gamma_n=\begin{cases} \gamma_{kn}, & \textit{if } kn=1 \textit{ or } p=0;\\ \beta_{kn,pn}, & \textit{if } kn\neq 1 \textit{ and } p=1,\ldots,k-1.\end{cases}$$

(2) *If k and n are arbitrary positive integers and $p\in\{0,\ldots,k-1\}$ then*

$$\alpha_{k,p}\delta_n=\begin{cases} \delta_{kn}, & \textit{if } kn=1 \textit{ or } p=k-1;\\ \beta_{kn,(p+1)n}, & \textit{if } kn\neq 1 \textit{ and } p=0,\ldots,k-2.\end{cases}$$

(3) *If k is an arbitrary positive integer $\geqslant 2$, n is an arbitrary positive integer, and $p\in\{1,\ldots,k-1\}$ then $\beta_{k,p}\gamma_n=\beta_{kn,pn}$*

(4) *If k is an arbitrary positive integer $\geqslant 2$, n is an arbitrary positive integer, and $p\in\{1,\ldots,k-1\}$ then*

$$\beta_{k,p}\delta_n=\begin{cases} \delta_{kn}, & \textit{if } p=k-1;\\ \beta_{kn,(p+1)n}, & \textit{if } p=1,\ldots,k-2.\end{cases}$$

(5) *If k and n are arbitrary positive integers and $p\in\{0,\ldots,k-1\}$ then $\gamma_n\alpha_{k,p}=\gamma_{nk}$.*

(6) *If k is an arbitrary positive integer $\geqslant 2$, n is an arbitrary positive integer, and $p\in\{1,\ldots,k-1\}$ then $\gamma_n\beta_{k,p}=\gamma_{nk}$.*

(7) *If k and n are arbitrary positive integers and $p\in\{0,\ldots,k-1\}$ then $\delta_n\alpha_{k,p}=\delta_{nk}$.*

(8) *If k is an arbitrary positive integer $\geqslant 2$, n is an arbitrary positive integer, and $p\in\{1,\ldots,k-1\}$ then $\delta_n\beta_{k,p}=\delta_{nk}$.*

Proof. (1) For any $i,j\in\omega$ we have that

$$(i,j,[0))\alpha_{k,p}\gamma_n=(ki,kj,[0))\gamma_n=(kni,knj,[0))$$

and

$$\begin{aligned}(i,j,[1))\alpha_{k,p}\gamma_n&=(p+ki,p+kj,[1))\gamma_n=\\&=(n(p+ki),n(p+kj),[0))=\\&=(pn+kni,pn+knj,[0)).\end{aligned}$$

The above equalities imply that

$$\alpha_{k,p}\gamma_n=\begin{cases} \gamma_{kn}, & \text{if } kn=1 \text{ or } p=0;\\ \beta_{kn,pn}, & \text{if } kn\neq 1 \text{ and } p=1,\ldots,k-1.\end{cases}$$

(2) For any $i,j\in\omega$ we have that

$$(i,j,[0))\alpha_{k,p}\delta_n=(ki,kj,[0))\delta_n=(kni,knj,[0))$$

and

$$\begin{aligned}(i,j,[1])\alpha_{k,p}\delta_n &= (p+ki,p+kj,[1])\delta_n = \\ &= (n(p+ki+1),n(p+kj+1),[0]) = \\ &= ((p+1)n+kni),(p+1)n+knj),[0]).\end{aligned}$$

The above equalities imply that

$$\alpha_{k,p}\delta_n = \begin{cases} \delta_{kn}, & \text{if } kn=1 \text{ or } p=k-1; \\ \beta_{kn,(p+1)n}, & \text{if } kn\neq 1 \text{ and } p=0,\ldots,k-2.\end{cases}$$

(3) For any $i,j\in\omega$ we have that

$$(i,j,[0])\beta_{k,p}\gamma_n = (ki,kj,[0])\gamma_n = (kni,knj,[0])$$

and

$$\begin{aligned}(i,j,[1])\beta_{k,p}\gamma_n &= (p+ki,p+kj,[0])\gamma_n = \\ &= (n(p+ki),n(p+kj),[0]) = \\ &= (pn+kni,pn+knj,[0]).\end{aligned}$$

Hence we get that $\beta_{k,p}\gamma_n = \beta_{kn,pn}$.

(4) For any $i,j\in\omega$ we have that

$$(i,j,[0])\beta_{k,p}\delta_n = (ki,kj,[0])\delta_n = (kni,knj,[0])$$

and

$$\begin{aligned}(i,j,[1])\beta_{k,p}\gamma_n &= (p+ki,p+kj,[0])\delta_n = \\ &= (n(p+ki+1),n(p+kj+1),[0]) = \\ &= ((p+1)n+kni,(p+1)n+knj,[0]).\end{aligned}$$

The above equalities imply that

$$\beta_{k,p}\delta_n = \begin{cases} \delta_{kn}, & \text{if } kn=1 \text{ or } p=k-1; \\ \beta_{kn,(p+1)n}, & \text{if } kn\neq 1 \text{ and } p=0,\ldots,k-2.\end{cases}$$

(5) For any $i,j\in\omega$ we have that

$$(i,j,[0])\gamma_n\alpha_{k,p} = (ni,nj,[0])\alpha_{k,p} = (nki,nkj,[0])$$

and

$$(i,j,[1])\gamma_n\alpha_{k,p} = (ni,nj,[0])\alpha_{k,p} = (nki,nkj,[0]).$$

This implies that $\gamma_n\alpha_{k,p} = \gamma_{nk}$.

(6) For any $i,j\in\omega$ we have that

$$(i,j,[0])\gamma_n\beta_{k,p} = (ni,nj,[0])\beta_{k,p} = (nki,nkj,[0])$$

and

$$(i,j,[1])\gamma_n\beta_{k,p} = (ni,nj,[0])\beta_{k,p} = (nki,nkj,[0]).$$

This implies that $\gamma_n\beta_{k,p} = \gamma_{nk}$.

(7) For any $i,j\in\omega$ we have that

$$(i,j,[0])\delta_n\alpha_{k,p} = (ni,nj,[0])\alpha_{k,p} = (nki,nkj,[0])$$

and

$$(i,j,[1])\delta_n\alpha_{k,p} = (n(i+1),n(j+1),[0])\alpha_{k,p} = (nk(i+1),nk(j+1),[0]).$$

The above equalities imply that $\delta_n\alpha_{k,p} = \delta_{nk}$.

The proof of statement (8) is similar to (7). □

We denote

$$\begin{aligned}
\langle\alpha\rangle &= \{\alpha_{k,p}\colon k\in\mathbb{N}, p\in\{0,\ldots,k-1\}\}\,;\\
\langle\beta\rangle &= \{\beta_{k,p}\colon k\in\mathbb{N}\setminus\{1\}, p\in\{1,\ldots,k-1\}\}\,;\\
\langle\gamma\rangle &= \{\gamma_k\colon k\in\mathbb{N}\}\,;\\
\langle\delta\rangle &= \{\delta_k\colon k\in\mathbb{N}\}\,;\\
\langle\alpha,\beta\rangle &= \{xy\colon x,y\in\langle\alpha\rangle\cup\langle\beta\rangle\}\,;\\
\langle\gamma,\delta\rangle &= \{xy\colon x,y\in\langle\gamma\rangle\cup\langle\delta\rangle\}\,;
\end{aligned}$$

Proposition 1. *Let $\mathscr{F}=\{[0],[1]\}$. Then the following statements hold.*

(1) *$\langle\alpha\rangle$, $\langle\beta\rangle$, $\langle\gamma\rangle$, and $\langle\delta\rangle$ are subsemigroups of $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$, and moreover $\langle\alpha\rangle$ is a cancellative submonoid of $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$, and the subsemigroups $\langle\gamma\rangle$ and $\langle\delta\rangle$ are isomorphic to the multiplicative semigroup of positive integers $\mathbb{N}_u$.*

(2) *$\mathbf{C}\langle\alpha\rangle = \boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})\setminus\langle\alpha\rangle$ is an ideal of $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$.*

(3) *$\langle\alpha,\beta\rangle = \boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ is a subsemigroup of $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$, and $\langle\beta\rangle$ is an ideal of $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$.*

(4) *The annihilating monoid endomorphism $\mathfrak{e}_{\mathbf{0}}$ of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$ is the zero of $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$.*

(5) *$\langle\gamma,\delta\rangle = \langle\gamma\rangle\cup\langle\delta\rangle = \boldsymbol{End}^*(\boldsymbol{B}_\omega^{\mathscr{F}})$ is a subsemigroup of $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$.*

(6) *$\langle\gamma,\delta\rangle\cup\{\mathfrak{e}_{\mathbf{0}}\}$, $\langle\gamma\rangle\cup\{\mathfrak{e}_{\mathbf{0}}\}$, and $\langle\delta\rangle\cup\{\mathfrak{e}_{\mathbf{0}}\}$ are right ideals of $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$, but they are not two-sided ideals of $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$.*

Proof. Statement (1) follows from Theorems 1 and 3.

Statement (2) follows from Theorem 1 and 4.

Statement (3) follows from Theorem 1.

Statement (4) is obvious.

Statements (5) and (6) follow from Theorems 3 and 4. □

Proposition 2 and Theorem 5 describe Green's relations on the monoid $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$.

Proposition 2. *Let $\mathscr{F}=\{[0],[1]\}$. Then the following statements hold.*

(1) *$\chi\mathscr{J}\alpha_{k_1,p_1}$ in $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$, $k_1\in\mathbb{N}$, $p_1\in\{0,\ldots,k_1-1\}$, if and only if $\chi=\alpha_{k_1,p_1}$.*

(2) *The relation $\beta_{k_1,p_1}\mathscr{L}\gamma_{n_1}$ in $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$ does not hold for any positive integers n_1 and $k_1\geqslant 2$, and any $p_1\in\{1,\ldots,k_1-1\}$.*

(3) *The relation $\beta_{k_1,p_1}\mathscr{L}\delta_{n_1}$ in $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$ does not hold for any positive integers n_1 and $k_1\geqslant 2$, and any $p_1\in\{1,\ldots,k_1-1\}$.*

(4) *$\beta_{k_1,p_1}\mathscr{L}\beta_{k_2,p_2}$ in $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$, $k_1,k_2\in\mathbb{N}\setminus\{1\}$, $p_1\in\{1,\ldots,k_1-1\}$, $p_2\in\{1,\ldots,k_2-1\}$, if and only if $k_1=k_2$ and $p_1=p_2$, i.e., $\beta_{k_1,p_1}=\beta_{k_2,p_2}$.*

(5) $\gamma_{n_1}\mathscr{L}\gamma_{n_2}$ *in* $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$, $n_1,n_2\in\mathbb{N}$, *if and only if* $n_1=n_2$, *i.e.*, $\gamma_{n_1}=\gamma_{n_2}$.
(6) $\gamma_{n_1}\mathscr{L}\delta_{n_2}$ *in* $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$, $n_1,n_2\in\mathbb{N}$, *if and only if* $n_1=n_2$.
(7) $\delta_{n_1}\mathscr{L}\delta_{n_2}$ *in* $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$, $n_1,n_2\in\mathbb{N}$, *if and only if* $n_1=n_2$, *i.e.*, $\delta_{n_1}=\delta_{n_2}$.
(8) *The relation* $\beta_{k_1,p_1}\mathscr{R}\gamma_{n_1}$ *in* $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ *does not hold for any positive integers* n_1 *and* $k_1\geqslant 2$, *and any* $p_1\in\{1,\ldots,k_1-1\}$.
(9) *The relation* $\beta_{k_1,p_1}\mathscr{R}\delta_{n_1}$ *in* $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ *does not hold for any positive integers* n_1 *and* $k_1\geqslant 2$, *and any* $p_1\in\{1,\ldots,k_1-1\}$.
(10) $\beta_{k_1,p_1}\mathscr{R}\beta_{k_2,p_2}$ *in* $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$, $k_1,k_2\in\mathbb{N}\setminus\{1\}$, $p_1\in\{1,\ldots,k_1-1\}$, $p_2\in\{1,\ldots,k_2-1\}$, *if and only if* $k_1=k_2$ *and* $p_1=p_2$, *i.e.*, $\beta_{k_1,p_1}=\beta_{k_2,p_2}$.
(11) $\gamma_{n_1}\mathscr{R}\gamma_{n_2}$ *in* $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$, $n_1,n_2\in\mathbb{N}$, *if and only if* $n_1=n_2$, *i.e.*, $\gamma_{n_1}=\gamma_{n_2}$.
(12) *The relation* $\gamma_n\mathscr{R}\delta_m$ *in* $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ *does not hold for any positive integers* n *and* m.
(13) $\delta_{n_1}\mathscr{R}\delta_{n_2}$ *in* $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$, $n_1,n_2\in\mathbb{N}$, *if and only if* $n_1=n_2$, *i.e.*, $\delta_{n_1}=\delta_{n_2}$.
(14) *The relation* $\beta_{k_1,p_1}\mathscr{J}\gamma_{n_1}$ *in* $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ *does not hold for any positive integers* n_1 *and* $k_1\geqslant 2$, *and any* $p_1\in\{1,\ldots,k_1-1\}$.
(15) *The relation* $\beta_{k_1,p_1}\mathscr{J}\delta_{n_1}$ *in* $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ *does not hold for any positive integers* n_1 *and* $k_1\geqslant 2$, *and any* $p_1\in\{1,\ldots,k_1-1\}$.

Proof. (1) ($\Rightarrow$) By Proposition 1(2) the set $\mathbf{C}\langle\alpha\rangle$ is an ideal of the monoid $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$, and hence $\chi\mathscr{J}\alpha_{k,p}$ in $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ if and only if $\chi\in\langle\alpha\rangle$.

Suppose that $\alpha_{k_1,p_1}\mathscr{J}\alpha_{k_2,p_2}$ in $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ for some positive integers k_1 and k_2, and $p_1\in\{0,\ldots,k_1-1\}$, $p_2\in\{0,\ldots,k_2-1\}$. The above arguments imply that there exist $k_1',k_1''\in\mathbb{N}$, $p_1'\in\{0,\ldots,k_1'-1\}$ and $p_1''\in\{0,\ldots,k_1''-1\}$ such that

$$\alpha_{k_1,p_1}=\alpha_{k_1',p_1'}\alpha_{k_2,p_2}\alpha_{k_1'',p_1''}.$$

By Theorem 1 we have that

$$\alpha_{k_1,p_1}=\alpha_{k_1',p_1'}\alpha_{k_2,p_2}\alpha_{k_1'',p_1''}=\alpha_{k_1',p_1'}\alpha_{k_2k_1'',p_1''+k_1''p_2}=\alpha_{k_1'k_2k_1'',p_1''+k_1''p_2+k_2k_1''p_1'}.$$

This implies that $k_1=k_1'k_2k_1''$ and $p_1=p_1''+k_1''p_2+k_2k_1''p_1'$. Similar calculations imply that if

$$\alpha_{k_2,p_2}=\alpha_{k_2',p_2'}\alpha_{k_1,p_1}\alpha_{k_2'',p_2''}$$

for some $k_2',k_2''\in\mathbb{N}$, $p_2'\in\{0,\ldots,k_2'-1\}$ and $p_2''\in\{0,\ldots,k_2''-1\}$, then $k_2=k_2'k_1k_2''$ and $p_2=p_2''+k_2''p_1+k_1k_2''p_2'$. Hence we have that $k_1|k_2$ and $k_2|k_1$. Thus we get that $k_1=k_2$ and hence $k_1'=k_2'=k_1''=k_2''=1$. The last equalities imply that $p_1'=p_2'=p_1''=p_2''=0$. Since $\alpha_{1,0}$ is the unit element of the monoid $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$, $\alpha_{k_1,p_1}=\alpha_{k_2,p_2}$.

The implications ($\Leftarrow$) is trivial.

(2) Suppose the contrary: the relation $\beta_{k_1,p_1}\mathscr{L}\gamma_{n_1}$ holds for some positive integers n_1 and $k_1\geqslant 2$, and $p_1\in\{1,\ldots,k_1-1\}$. Then there exist $\varepsilon_1,\varepsilon_2\in\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ such that $\beta_{k_1,p_1}=\varepsilon_1\gamma_{n_1}$ and $\gamma_{n_1}=\varepsilon_2\beta_{k_1,p_1}$.

Theorem 4 implies that the equality $\beta_{k_1,p_1}=\varepsilon_1\gamma_{n_1}$ holds only in one of the following two cases:

(1_1) $\varepsilon_1=\alpha_{k,p}$ for some positive integer k and $p\in\{0,\ldots,k-1\}$;
(2_1) $\varepsilon_1=\beta_{k,p}$ for some positive integer $k\geqslant 2$ and $p\in\{1,\ldots,k-1\}$.

Theorems 1 and 4 imply that the equality $\gamma_{n_1} = \varepsilon_2\beta_{k_1,p_1}$ holds in the case when $\varepsilon_2 = \gamma_m$ for some positive integer m. By Theorem 4(6) we have that

$$\gamma_{n_1} = \gamma_m\beta_{k_1,p_1} = \gamma_{mk_1}.$$

Then in case (1_1) by Theorem 4(1) we get that

$$\beta_{k_1,p_1} = \alpha_{k,p}\gamma_{n_1} = \alpha_{k,p}\gamma_{mk_1} = \beta_{kmk_1,pmk_1},$$

because $k_1 \geqslant 2$. The definition of the endomorphism β_{k_1,p_1} implies that $k_1 = kmk_1$ and $p_1 = pmk_1$. By the first equality we get that $km = 1$ and hence $k = m = 1$. Then the definition of the endomorphism $\alpha_{k,p}$ implies that $p = 0$ and hence $p_1 = pmk_1 = 0$. But the equality $p_1 = 0$ contradicts the definition of the endomorphism β_{k_1,p_1}, because $p_1 \in \{1, \ldots, k_1 - 1\}$. The obtained contradiction implies that case (1_1) does not hold.

In case (2_1) by Theorem 4(3) we get that

$$\beta_{k_1,p_1} = \beta_{k,p}\gamma_{n_1} = \beta_{k,p}\gamma_{mk_1} = \beta_{kmk_1,pmk_1}.$$

The definition of the endomorphism β_{k_1,p_1} implies that $k_1 = kmk_1$ and $p_1 = pmk_1$. Next, by the similar way as in the previous case we show that case (2_1) does not hold. This completes the proof of the statement.

(3) Suppose to the contrary that the relation $\beta_{k_1,p_1}\mathscr{L}\delta_{n_1}$ holds for some positive integers n_1 and $k_1 \geqslant 2$, and $p_1 \in \{1, \ldots, k_1 - 1\}$. Then there exist $\varepsilon_1, \varepsilon_2 \in \boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ such that $\beta_{k_1,p_1} = \varepsilon_1\delta_{n_1}$ and $\delta_{n_1} = \varepsilon_2\beta_{k_1,p_1}$.

Theorem 4 implies that the equality $\beta_{k_1,p_1} = \varepsilon_1\delta_{n_1}$ holds only in one of the following two cases:

(1_2) $\varepsilon_1 = \alpha_{k,p}$ for some positive integer k and $p \in \{0, \ldots, k - 1\}$;
(2_2) $\varepsilon_1 = \beta_{k,p}$ for some positive integer $k \geqslant 2$ and $p \in \{1, \ldots, k - 1\}$.

Theorem 4 implies that the equality $\delta_{n_1} = \varepsilon_2\beta_{k_1,p_1}$ holds in the case when $\varepsilon_2 = \delta_m$ for some positive integer m. By Theorem 4(8) we have that

$$\delta_{n_1} = \delta_m\beta_{k_1,p_1} = \delta_{mk_1}.$$

Then in case (1_2) by Theorem 4(2) we have that

$$\beta_{k_1,p_1} = \alpha_{k,p}\delta_{n_1} = \alpha_{k,p}\delta_{mk_1} = \beta_{kmk_1,(p+1)mk_1},$$

because $k_1 \geqslant 2$. The definition of the endomorphism β_{k_1,p_1} implies that $k_1 = kmk_1$ and $p_1 = (p+1)mk_1$. By the first equality we get that $km = 1$ and hence $k = m = 1$. Then the definition of the endomorphism $\alpha_{k,p}$ implies that $p = 0$ which implies that $p_1 = mk_1 = k_1$. But by the definition of the endomorphism β_{k_1,p_1} we have that $k_1 > p_1$, a contradiction. The obtained contradiction implies that case (1_2) does not hold.

In case (2_2) by Theorem 4(4) we get that

$$\beta_{k_1,p_1} = \beta_{k,p}\delta_{n_1} = \beta_{k,p}\delta_{mk_1} = \beta_{kmk_1,(p+1)mk_1}.$$

The definition of the endomorphism β_{k_1,p_1} implies that $k_1 = kmk_1$ and $p_1 = (p+1)mk_1$. Next, by the similar way as in the previous case we show that case (2_2) does not hold. This completes the proof of the statement.

(4) ($\Rightarrow$) Suppose that $\beta_{k_1,p_1}\mathscr{L}\beta_{k_2,p_2}$ in $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ for some positive integers $k_1 \geqslant 2$, $k_2 \geqslant 2$, $p_1 \in \{1,\ldots,k_1-1\}$, and $p_2 \in \{1,\ldots,k_2-1\}$. Then there exist $\varepsilon_1, \varepsilon_2 \in \boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ such that

$$\beta_{k_1,p_1} = \varepsilon_1\beta_{k_2,p_2} \qquad \text{and} \qquad \beta_{k_2,p_2} = \varepsilon_2\beta_{k_1,p_1}. \tag{3}$$

By statements (6) and (8) of Theorem 4 we have that equalities (3) hold when $\varepsilon_1, \varepsilon_2 \in \langle\alpha\rangle \cup \langle\beta\rangle$. Then the proof of the requested statement is similar to Proposition 3.3(4) of [8].

The implications ($\Leftarrow$) is trivial.

(5) ($\Rightarrow$) Suppose that $\gamma_{n_1}\mathscr{L}\gamma_{n_2}$ in $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ for some positive integers n_1 and n_2. Then there exist $\varepsilon_1, \varepsilon_2 \in \boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ such that

$$\gamma_{n_1} = \varepsilon_1\gamma_{n_2} \qquad \text{and} \qquad \gamma_{n_2} = \varepsilon_2\gamma_{n_1}.$$

By Theorem 3(3) we have that $\varepsilon_1 \neq \delta_{n'}$ and $\varepsilon_2 \neq \delta_{n''}$ for any positive integers n' and n''. Also, by Theorem 4(3), $\varepsilon_1 \neq \beta_{k',p'}$ and $\varepsilon_2 \neq \beta_{k'',p''}$ for any positive integers $k' \geqslant 2$ and $k'' \geqslant 2$, $p' \in \{1,\ldots,k'-1\}$, and $p'' \in \{1,\ldots,k''-1\}$.

Suppose that $\varepsilon_1 = \alpha_{k,p}$ for some positive integer k and $p \in \{0,\ldots,k\}$. Then by Theorem 4(1) the equality $\gamma_{n_1} = \alpha_{k,p}\gamma_{n_2}$ holds in the case when $kn_2 = 1$ or $p = 0$.

If $kn_2 = 1$ then $k = n_2 = 1$ which implies that $p = 0$, and hence

$$\gamma_{n_1} = \alpha_{1,0}\gamma_{n_2} = \gamma_{n_2}.$$

If $p = 0$ then

$$\gamma_{n_1} = \alpha_{k,0}\gamma_{n_2} = \gamma_{kn_2}.$$

In this case we have that either $\varepsilon_2 = \alpha_{k',p'}$ or $\varepsilon_2 = \gamma_{n'}$. If $\varepsilon_2 = \alpha_{k',p'}$ then by Theorem 4(1),

$$\gamma_{n_2} = \alpha_{k',p'}\gamma_{n_1} = \gamma_{k'n_1}$$

and $k'n_1 = 1$ or $p' = 0$. If $k'n_1 = 1$ then $k' = n_1 = 1$ and $p' = 0$. Then

$$\gamma_{n_2} = \alpha_{1,0}\gamma_{n_1} = \gamma_{n_1}.$$

If $p' = 0$ then we have that

$$\gamma_{n_2} = \alpha_{k',0}\gamma_{n_1} = \gamma_{k'n_1}.$$

Thus, in the case when $\varepsilon_1 = \alpha_{k,p}$ and $\varepsilon_2 = \alpha_{k',p'}$ by the definition of the element γ_n of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ we get that

$$n_2 = k'n_1 = k'kn_2,$$

which implies that $k' = k = 1$, and hence $n_1 = n_2$.

In the case when $\varepsilon_1 = \alpha_{k,p}$ and $\varepsilon_2 = \gamma_{n'}$ we have that

$$\gamma_{n_1} = \alpha_{k,0}\gamma_{n_2} = \alpha_{k,0}\gamma_{n'}\gamma_{n_1} = \alpha_{k,0}\gamma_{n'n_1} = \gamma_{kn'n_1},$$

which implies that $kn' = 1$, because k, n' and n_1 are positive integers, and hence $k = n' = 1$. Then

$$\gamma_{n_2} = \gamma_{n'}\gamma_{n_1} = \gamma_{n_1}.$$

In the case when $\varepsilon_1 = \gamma_{n'}$ and $\varepsilon_2 = \alpha_{k,p}$ by the similar way as in the above case we obtain that $\gamma_{n_1} = \gamma_{n_2}$.

In the case when $\varepsilon_1 = \gamma_{n_1'}$ and $\varepsilon_2 = \gamma_{n_2'}$ by Theorem 3(1) we get that

$$\gamma_{n_1} = \gamma_{n_1'}\gamma_{n_2} = \gamma_{n_1'n_2} \qquad \text{and} \qquad \gamma_{n_2} = \gamma_{n_2'}\gamma_{n_1} = \gamma_{n_2'n_1}.$$

Then by the definition of the element γ_n of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ we get that

$$n_1 = n_1' n_2 = n_1' n_2' n_1.$$

Since n_1, n_2, n_1', and n_2' are positive integers, $n_1' n_2' = 1$, and hence $n_1' = n_2' = 1$. This implies that $n_1 = n_2$.

The implications $(\Leftarrow)$ is trivial.

(6) $(\Rightarrow)$ Suppose that $\gamma_{n_1} \mathscr{L} \delta_{n_2}$ in $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ for some positive integers n_1 and n_2. Then there exist $\varepsilon_1, \varepsilon_2 \in \boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ such that

$$\gamma_{n_1} = \varepsilon_1 \delta_{n_2} \qquad \text{and} \qquad \delta_{n_2} = \varepsilon_2 \gamma_{n_1}.$$

The equality $\gamma_{n_1} = \varepsilon_1 \delta_{n_2}$ and Theorems 3 and 4 imply that $\varepsilon_1 = \gamma_m$ for some positive integer m. By Theorem 3(2) we have that

$$\gamma_{n_1} = \gamma_m \delta_{n_2} = \gamma_{m n_2},$$

and hence by the definition of the element γ_n of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ we get that $n_1 = m n_2$. Also, the equality $\delta_{n_2} = \varepsilon_2 \gamma_{n_1}$ and Theorems 3 and 4 imply that $\varepsilon_2 = \delta_{m'}$ for some positive integer m'. By Theorem 3(3) we have that

$$\delta_{n_2} = \delta_{m'} \gamma_{n_1} = \delta_{m' n_1},$$

and hence by the definition of the element δ_n of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ we get that $n_2 = m' n_1$. Then

$$n_1 = m n_2 = m m' n_1$$

for some positive integers m and m'. This implies that $m m' = 1$, and hence $m = m' = 1$. Thus, we obtain that $n_1 = n_2$.

The implications $(\Leftarrow)$ is trivial.

(7) $(\Rightarrow)$ Suppose that $\delta_{n_1} \mathscr{L} \delta_{n_2}$ in $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ for some positive integers n_1 and n_2. Then there exist $\varepsilon_1, \varepsilon_2 \in \boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ such that

$$\delta_{n_1} = \varepsilon_1 \delta_{n_2} \qquad \text{and} \qquad \delta_{n_2} = \varepsilon_2 \delta_{n_1}.$$

Then by Theorems 3 and 4 for the element ε_1 one of the following cases holds

(1_3) $\varepsilon_1 = \alpha_{k,p}$ for some positive integer k and $p \in \{0, \ldots, k-1\}$;
(2_3) $\varepsilon_1 = \beta_{k,p}$ for some positive integer $k \geqslant 2$ and $p \in \{1, \ldots, k-1\}$;
(3_3) $\varepsilon_1 = \delta_k$ for some positive integer k,

and for the element ε_2 one of the following cases holds

(1_4) $\varepsilon_2 = \alpha_{k',p'}$ for some positive integer k' and $p' \in \{0, \ldots, k'-1\}$;
(2_4) $\varepsilon_2 = \beta_{k',p'}$ for some positive integer $k' \geqslant 2$ and $p' \in \{1, \ldots, k'-1\}$;
(3_4) $\varepsilon_2 = \delta_{k'}$ for some positive integer k'.

Suppose case (1_3) holds. By Theorems 4(2) we have that

$$\delta_{n_1} = \alpha_{k,p} \delta_{n_2} = \delta_{k n_2}$$

in the case when $k n_2 = 1$ or $p = k-1$. If $k n_2 = 1$ then by the definition of the element δ_n of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ we have that $n_1 = n_2 = k = 1$ and $p = 0$, and hence the requested statement holds.

Suppose that $p = k-1$ for some positive integer $k \geqslant 2$. If case (1_4) holds, i.e.,

$$\delta_{n_2} = \alpha_{k',p'} \delta_{n_1} = \delta_{k' n_1},$$

then $k'n_1 \neq 1$, otherwise in this case we have $n_1 = n_2 = 1$, which implies that $k = 1$. Hence we suppose that $p' = k' - 1$ for some positive integer $k' \geqslant 2$. Then we get that $\delta_{n_2} = \delta_{k'n_1}$ and $\delta_{n_1} = \delta_{kn_2}$, and by the definition of the element δ_n of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ we obtain that $n_2 = k'n_1 = k'kn_2$, and hence $k'k = 1$ which contradicts our assumptions that $k \geqslant 2$ and $k' \geqslant 2$.

Next, suppose that $k \geqslant 2$ and case (2_4) holds, i.e.,

$$\delta_{n_2} = \beta_{k',p'}\delta_{n_1} = \delta_{k'n_1} \qquad \text{and} \qquad \delta_{n_1} = \delta_{kn_2}.$$

Then similar as in previous case we get that $n_2 = k'n_1 = k'kn_2$. This and the definition of the element δ_n of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ contradict the condition $k' \geqslant 2$.

Suppose that $k \geqslant 2$ and case (3_4) holds. By Theorem 3(4) we have that

$$\delta_{n_2} = \delta_{k'}\delta_{n_1} = \delta_{k'n_1} \qquad \text{and} \qquad \delta_{n_1} = \delta_{kn_2},$$

and hence by the definition of the element δ_n of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ we get that $n_2 = k'n_1 = k'kn_2$. This implies that $k' = k = 1$, and hence $n_1 = n_2$.

Suppose that case (2_3) holds. If in addition case (1_4) holds then the proof of the equality $n_1 = n_2$ is similar to the case when conditions (1_3) and (2_4) hold together.

Suppose that cases (2_3) and (2_4) hold together, i.e., $\varepsilon_1 = \beta_{k,p}$ and $\varepsilon_2 = \beta_{k',p'}$ for some positive integers $k' \geqslant 2$ and $k \geqslant 2$, $p \in \{1, \ldots, k-1\}$ and $p' \in \{1, \ldots, k'-1\}$. Theorem 4(4) implies that

$$\delta_{n_1} = \beta_{k,p}\delta_{n_2} = \delta_{kn_2} \qquad \text{and} \qquad \delta_{n_2} = \beta_{k',p'}\delta_{n_1} = \delta_{k'n_1},$$

and here $k \geqslant 2$ and $k' \geqslant 2$. Then the definition of the element δ_n of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ implies that

$$n_1 = kn_2 = kk'n_1,$$

and hence $kk' = 1$, which contradicts any of conditions $k \geqslant 2$ and $k' \geqslant 2$. The obtained contradiction implies that (2_3) and (2_4) do not hold together.

Suppose that cases (2_3) and (3_4) hold together, i.e., $\varepsilon_1 = \beta_{k,p}$ and $\varepsilon_2 = \delta_{k'}$ for some positive integers k' and $k \geqslant 2$, and $p \in \{1, \ldots, k-1\}$. Then Theorem 4(4) and Theorem 3(4) imply that

$$\delta_{n_1} = \beta_{k,p}\delta_{n_2} = \delta_{kn_2} \qquad \text{and} \qquad \delta_{n_2} = \delta_{k'}\delta_{n_1} = \delta_{k'n_1},$$

and here $k \geqslant 2$. The definition of the element δ_n of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ implies that

$$n_1 = kn_2 = kk'n_1,$$

and hence $kk' = 1$, which contradicts any of conditions $k \geqslant 2$. The obtained contradiction implies that (2_3) and (3_4) do not hold together.

Now, we suppose that case (3_3) holds. If case (1_4) holds as well, then the proof is similar to the case when (1_3) and (3_4) hold. Also, if cases (3_3) and (2_4) hold together, then the proof is similar to the case when (2_3) and (3_4) hold.

Suppose that cases (3_3) and (3_4) hold together, i.e., $\varepsilon_1 = \delta_k$ and $\varepsilon_2 = \delta_{k'}$ for some positive integers k and k'. Theorem 3(4) imply that

$$\delta_{n_1} = \delta_k\delta_{n_2} = \delta_{kn_2} \qquad \text{and} \qquad \delta_{n_2} = \delta_{k'}\delta_{n_1} = \delta_{k'n_1}.$$

The definition of the element δ_n of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ implies that

$$n_1 = kn_2 = kk'n_1,$$

and hence $k = k' = 1$, which implies that $n_1 = n_2$.

The proof in other cases are similar to the above considered cases.

The implications ($\Leftarrow$) is trivial.

(8) By Theorems 3 and 4 there exists no $\varepsilon_1 \in \boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ such that $\beta_{k_1,p_1} = \gamma_{n_1}\varepsilon_1$, and hence the requested statement holds.

(9) By Theorems 3 and 4 there exists no $\varepsilon_1 \in \boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ such that $\beta_{k_1,p_1} = \delta_{n_1}\varepsilon_1$, and hence the requested statement holds.

(10) ($\Rightarrow$) Suppose that $\beta_{k_1,p_1}\mathscr{R}\beta_{k_2,p_2}$ in $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ for some positive integers $k_1 \geqslant 2$, $k_2 \geqslant 2$, $p_1 \in \{1,\ldots,k_1-1\}$, and $p_2 \in \{1,\ldots,k_2-1\}$. Then there exist $\varepsilon_1, \varepsilon_2 \in \boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ such that

$$\beta_{k_1,p_1} = \beta_{k_2,p_2}\varepsilon_1 \qquad \text{and} \qquad \beta_{k_2,p_2} = \beta_{k_1,p_1}\varepsilon_2.$$

By Theorems 1(6) and 4(3)–(4) for the element ε_1 one of the following cases holds

(1_5) $\varepsilon_1 = \alpha_{k,p}$ for some positive integer k and $p \in \{0,\ldots,k-1\}$;
(2_5) $\varepsilon_1 = \gamma_n$ for some positive integer n;
(3_5) $\varepsilon_1 = \delta_n$ for some positive integer n,

and for the element ε_2 one of the following cases holds

(1_6) $\varepsilon_2 = \alpha_{k',p'}$ for some positive integer k' and $p' \in \{0,\ldots,k'-1\}$;
(2_6) $\varepsilon_2 = \gamma_{n'}$ for some positive integer n';
(3_6) $\varepsilon_2 = \delta_{n'}$ for some positive integer n'.

Suppose that cases (1_5) and (1_6) hold together. By Theorem 1(6) we have that

$$\beta_{k_1,p_1} = \beta_{k_2,p_2}\alpha_{k,p} = \beta_{k_2k,kp_2} \qquad \text{and} \qquad \beta_{k_2,p_2} = \beta_{k_1,p_1}\alpha_{k',p'} = \beta_{k_1k',k'p_1}. \tag{4}$$

By the definition of the element $\beta_{s,t}$ of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ we obtain that

$$k_1 = k_2k = k_1k'k \qquad \text{and} \qquad p_1 = kp_2 = kk'p_1.$$

Hence we have that $k = k' = 1$, because k and k' are positive integers. This implies that $k_1 = k_2$ and $p_1 = p_2$.

If cases (1_5) and (2_6) hold together then by Theorem 4(3) we get that

$$\beta_{k_2,p_2} = \beta_{k_1,p_1}\gamma_{n'} = \beta_{k_1n',p_1n'}.$$

The above equalities, the first equality of (4), and the definition of the element $\beta_{s,t}$ of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ imply that

$$k_2 = k_1n' = k_2kn',$$

and hence $k = n' = 1$, because k and n' are positive integers. This implies that $p_2 = p_1$.

If cases (1_5) and (3_6) hold together then by Theorem 4(4) we get that

$$\beta_{k_2,p_2} = \beta_{k_1,p_1}\delta_{n'} = \beta_{k_1n',(p_1+1)n'}.$$

in the case when $p_1 \in \{1,\ldots,k_1-2\}$. Then the above equalities, the first equality of (4), and the definition of the element $\beta_{s,t}$ of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ imply that

$$k_2 = k_1n' = k_2kn',$$

and hence $k = n' = 1$, because k and n' are positive integers. Hence we get that

$$p_1 = kp_2 = k(p_1+1)n' = p_1+1,$$

a contradiction. The obtained contradiction implies that cases (1_5) and (3_6) do not hold together.

Suppose that cases (2_5) and (2_6) hold together. By Theorem 4(3) we have that

$$\beta_{k_1,p_1} = \beta_{k_2,p_2}\gamma_n = \beta_{k_2n,p_2n} \qquad \text{and} \qquad \beta_{k_2,p_2} = \beta_{k_1,p_1}\gamma_{n'} = \beta_{k_1n',p_1n'}.$$

The definition of the element $\beta_{s,t}$ of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ imply that

$$k_1 = k_2n = k_1n'n,$$

and hence $n = n' = 1$, because n and n' are positive integers. This implies that $k_1 = k_2$ and $p_1 = p_2$.

Suppose that cases (2_5) and (3_6) hold together. By statements (3) and (4) of Theorem 4 we have that

$$\beta_{k_1,p_1} = \beta_{k_2,p_2}\gamma_n = \beta_{k_2n,p_2n} \qquad \text{and} \qquad \beta_{k_2,p_2} = \beta_{k_1,p_1}\delta_{n'} = \beta_{k_1n',(p_1+1)n'},$$

and hence by the definition of the element $\beta_{s,t}$ of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ we get that

$$k_1 = k_2n = k_1n'n \qquad \text{and} \qquad p_1 = p_2n = (p_1+1)n'n.$$

Since n and n' are positive integers, $n = n' = 1$, and hence $p_1 = p_1 + 1$, a contradiction. The obtained contradiction implies that cases (2_5) and (3_6) do not hold together.

Suppose that cases (3_5) and (3_6) hold together. By Theorem 4(4) we have that

$$\beta_{k_1,p_1} = \beta_{k_2,p_2}\delta_n = \beta_{k_2n,(p_2+1)n} \qquad \text{and} \qquad \beta_{k_2,p_2} = \beta_{k_1,p_1}\delta_{n'} = \beta_{k_1n',(p_1+1)n'},$$

and hence by the definition of the element $\beta_{s,t}$ of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ we get that

$$k_1 = k_2n = k_1n'n \qquad \text{and} \qquad p_1 = (p_2+1)n = ((p_1+1)n'+1)n.$$

Since n and n' are positive integers, $n = n' = 1$, and hence $p_1 = p_1 + 2$, a contradiction. The obtained contradiction implies that cases (3_5) and (3_6) do not hold together.

The proofs in other cases are similar to the above considered cases.

The implication ($\Leftarrow$) is trivial.

(11) ($\Rightarrow$) Suppose that $\gamma_{n_1}\mathscr{R}\gamma_{n_2}$ in $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ for some positive integers n_1 and n_2. Then there exist $\varepsilon_1, \varepsilon_2 \in \boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ such that

$$\gamma_{n_1} = \gamma_{n_2}\varepsilon_1 \qquad \text{and} \qquad \gamma_{n_2} = \gamma_{n_1}\varepsilon_2.$$

By Theorems 3 and 4 for the element ε_1 one of the following cases holds

(1_7) $\varepsilon_1 = \alpha_{k,p}$ for some positive integer k and $p \in \{0, \ldots, k-1\}$;
(2_7) $\varepsilon_1 = \beta_{k,p}$ for some positive integer $k \geqslant 2$ and $p \in \{1, \ldots, k-1\}$;
(3_7) $\varepsilon_1 = \gamma_n$ for some positive integer n;
(4_7) $\varepsilon_1 = \delta_n$ for some positive integer n,

and for the element ε_2 one of the following cases holds

(1_8) $\varepsilon_2 = \alpha_{k',p'}$ for some positive integer k' and $p' \in \{0, \ldots, k'-1\}$;
(2_8) $\varepsilon_2 = \beta_{k',p'}$ for some positive integer $k' \geqslant 2$ and $p' \in \{1, \ldots, k'-1\}$;
(3_8) $\varepsilon_2 = \gamma_{n'}$ for some positive integer n';
(4_8) $\varepsilon_2 = \delta_{n'}$ for some positive integer n'.

If $i,j\in\{1,2\}$ then by the corresponding statements of Theorem 4 we have that

$$\gamma_{n_1}=\gamma_{n_2}\varepsilon_1=\gamma_{n_2k} \qquad \text{and} \qquad \gamma_{n_2}=\beta_{n_1}\varepsilon_2=\gamma_{n_1k'}.$$

The definition of the element γ_m of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ implies that $n_1=n_2k=n_1kk'$, and hence $k=k'=1$, because k and k' are positive integers. This implies that $n_1=n_2$.

If $i\in\{1,2\}$ and $j\in\{3,4\}$ then by the corresponding statements of Theorems 3 and 4 we have that

$$\gamma_{n_1}=\gamma_{n_2}\varepsilon_1=\gamma_{n_2k} \qquad \text{and} \qquad \gamma_{n_2}=\beta_{n_1}\varepsilon_2=\gamma_{n_1n'}.$$

Then the definition of the element γ_m of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ implies that $n_1=n_2k=n_1kn'$, and hence $k=n'=1$, because k and n' are positive integers. This implies that $n_1=n_2$.

In the case of $i\in\{3,4\}$ and $j\in\{1,2\}$ the proof is similar to the above case.

If $i,j\in\{3,4\}$ then by the corresponding statements of Theorem 3 we have that

$$\gamma_{n_1}=\gamma_{n_2}\varepsilon_1=\gamma_{n_2n} \qquad \text{and} \qquad \gamma_{n_2}=\beta_{n_1}\varepsilon_2=\gamma_{n_1n'}.$$

Then the definition of the element γ_m of $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ implies that $n_1=n_2n=n_1nn'$, and hence $n=n'=1$, because n and n' are positive integers. This implies that $n_1=n_2$.

Implications $(\Leftarrow)$ is trivial.

(12) By Theorems 3 and 4 there exists no $\varepsilon_1\in\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ such that $\gamma_n=\delta_m\varepsilon_1$, $m,n\in\mathbb{N}$, and hence the requested statement holds.

(13) $(\Rightarrow)$ Suppose that $\delta_{n_1}\mathscr{R}\delta_{n_2}$ in $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ for some positive integers n_1 and n_2. Then there exist $\varepsilon_1,\varepsilon_2\in\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ such that

$$\delta_{n_1}=\delta_{n_2}\varepsilon_1 \qquad \text{and} \qquad \delta_{n_2}=\delta_{n_1}\varepsilon_2.$$

By Theorems 3 and 4 for the element ε_1 one of the following cases holds

(1_9) $\varepsilon_1=\alpha_{k,p}$ for some positive integer k and $p\in\{0,\ldots,k-1\}$;
(2_9) $\varepsilon_1=\beta_{k,p}$ for some positive integer $k\geqslant 2$ and $p\in\{1,\ldots,k-1\}$;
(3_9) $\varepsilon_1=\gamma_n$ for some positive integer n;
(4_9) $\varepsilon_1=\delta_n$ for some positive integer n,

and for the element ε_2 one of the following cases holds

(1_{10}) $\varepsilon_2=\alpha_{k',p'}$ for some positive integer k' and $p'\in\{0,\ldots,k'-1\}$;
(2_{10}) $\varepsilon_2=\beta_{k',p'}$ for some positive integer $k'\geqslant 2$ and $p'\in\{1,\ldots,k'-1\}$;
(3_{10}) $\varepsilon_2=\gamma_{n'}$ for some positive integer n';
(4_{10}) $\varepsilon_2=\delta_{n'}$ for some positive integer n'.

Next, the proof of the statement $n_1=n_2$ word by word repeats the proof of statement (11).

(14) Suppose to the contrary that there exist positive integers n_1 and $k_1\geqslant 2$, and $p_1\in\{1,\ldots,k_1-1\}$ such that the relation $\beta_{k_1,p_1}\mathscr{J}\gamma_{n_1}$ holds in $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$. Then by the definition of the relation $\mathscr{J}$ there exist $\varepsilon_{11},\varepsilon_{12},\varepsilon_{21},\varepsilon_{22}\in\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})^1$ such that $\beta_{k_1,p_1}=\varepsilon_{11}\gamma_{n_1}\varepsilon_{12}$ and $\gamma_{n_1}=\varepsilon_{21}\beta_{k_1,p_1}\varepsilon_{22}$.

By Theorem 1(4) we have that $\langle\alpha\rangle\cdot\langle\beta\rangle\cup\langle\beta\rangle\cdot\langle\alpha\rangle\subseteq\langle\beta\rangle$, and hence by Theorems 4 and 1 we obtain that $\varepsilon_{21}\in\langle\gamma\rangle$. Also, Theorem 3 implies that $\varepsilon_{11}\notin\langle\gamma\rangle\cup\langle\delta\rangle$. Then we have that

$$\beta_{k_1,p_1}=\varepsilon_{11}\gamma_{n_1}\varepsilon_{12}=\varepsilon_{11}\gamma_{k_4}\beta_{k_1,p_1}\varepsilon_{22}\varepsilon_{12}.$$

Later we consider the following cases.

(1_{14}) If $\varepsilon_{11}=\alpha_{k_2,p_2}$, $\varepsilon_{12}=\alpha_{k_3,p_3}$, and $\varepsilon_{22}=\alpha_{k_5,p_5}$, then by Theorems 1 and 4 we have that

$$\begin{aligned}
\beta_{k_1,p_1} &= \alpha_{k_2,p_2}\gamma_{k_4}\beta_{k_1,p_1}\alpha_{k_5,p_5}\alpha_{k_3,p_3} = \\
&= \alpha_{k_2,p_2}\gamma_{k_4k_1}\alpha_{k_5,p_5}\alpha_{k_3,p_3} = \\
&= \alpha_{k_2,p_2}\gamma_{k_4k_1k_5}\alpha_{k_3,p_3} = \\
&= \alpha_{k_2,p_2}\gamma_{k_4k_1k_5k_3} = \\
&= \begin{cases} \gamma_{k_2k_4k_1k_5k_3}, & \text{if } k_3k_4k_1k_5k_3=1 \text{ or } p=0; \\ \beta_{k_2k_4k_1k_5k_3,p_2k_4k_1k_5k_3}, & \text{if } k_3k_4k_1k_5k_3\neq 1 \text{ and } p=1,\ldots,k_3-1. \end{cases}
\end{aligned}$$

The first case is impossible. In the second case we have that $k_2k_4k_1k_5k_3=k_1$ which implies that $k_2=1$. Hence we have that $p_2=0$, a contradiction. The obtained contradiction implies that case (1_{14}) does not hold.

(2_{14}) If $\varepsilon_{11}=\alpha_{k_2,p_2}$, $\varepsilon_{12}=\alpha_{k_3,p_3}$, and $\varepsilon_{22}\in\{\beta_{k_5,p_5},\gamma_{k_5},\delta_{k_5}\}$, then by Theorems 1 and 4 and by the similar calculations as in case (1_{14}) we get that

$$\begin{aligned}
\beta_{k_1,p_1} &= \alpha_{k_2,p_2}\gamma_{k_4}\beta_{k_1,p_1}\varepsilon_{22}\alpha_{k_3,p_3} = \\
&= \begin{cases} \gamma_{k_2k_4k_1k_5k_3}, & \text{if } k_3k_4k_1k_5k_3=1 \text{ or } p=0; \\ \beta_{k_2k_4k_1k_5k_3,p_2k_4k_1k_5k_3}, & \text{if } k_3k_4k_1k_5k_3\neq 1 \text{ and } p=1,\ldots,k_3-1. \end{cases}
\end{aligned}$$

As in the previous case we obtain that case (2_{14}) does not hold.

(3_{14}) If $\varepsilon_{11}=\alpha_{k_2,p_2}$, $\varepsilon_{12}\in\{\beta_{k_3,p_3},\gamma_{k_3},\delta_{k_3}\}$, and $\varepsilon_{22}\in\{\alpha_{k_5,p_5},\beta_{k_5,p_5},\gamma_{k_5},\delta_{k_5}\}$, then by Theorems 1 and 4 and by the similar calculations as in case (1_{14}) we get that

$$\begin{aligned}
\beta_{k_1,p_1} &= \alpha_{k_2,p_2}\gamma_{k_4}\beta_{k_1,p_1}\varepsilon_{22}\varepsilon_{12} = \\
&= \begin{cases} \gamma_{k_2k_4k_1k_5k_3}, & \text{if } k_3k_4k_1k_5k_3=1 \text{ or } p=0; \\ \beta_{k_2k_4k_1k_5k_3,p_2k_4k_1k_5k_3}, & \text{if } k_3k_4k_1k_5k_3\neq 1 \text{ and } p=1,\ldots,k_3-1. \end{cases}
\end{aligned}$$

As in case (1_{14}) we obtain that case (3_{14}) does not hold.

(4_{14}) If $\varepsilon_{11}=\beta_{k_2,p_2}$, $\varepsilon_{12}=\alpha_{k_3,p_3}$, and $\varepsilon_{22}=\alpha_{k_5,p_5}$, then by Theorems 1, 3, and 4 we have that

$$\begin{aligned}
\beta_{k_1,p_1} &= \beta_{k_2,p_2}\gamma_{k_4}\beta_{k_1,p_1}\alpha_{k_5,p_5}\alpha_{k_3,p_3} = \\
&= \beta_{k_2,p_2}\gamma_{k_4k_1}\alpha_{k_5,p_5}\alpha_{k_3,p_3} = \\
&= \beta_{k_2,p_2}\gamma_{k_4k_1k_5}\alpha_{k_3,p_3} = \\
&= \beta_{k_2,p_2}\gamma_{k_4k_1k_5k_3} = \\
&= \beta_{k_2k_4k_1k_5k_3,p_2k_4k_1k_5k_3},
\end{aligned}$$

which implies that $k_2k_4k_1k_5k_3=k_1$. Hence $k_2=1$ which contradicts the definition of the element β_{k_2,p_2}. The obtained contradiction implies that case (4_{14}) does not hold.

(5_{14}) If $\varepsilon_{11}=\beta_{k_2,p_2}$, $\varepsilon_{12}=\alpha_{k_3,p_3}$, and $\varepsilon_{22}\in\{\beta_{k_5,p_5},\gamma_{k_5},\delta_{k_5}\}$, then by Theorems 1, 3, and 4 and by the similar calculations as in case (4_{14}) we get that

$$\begin{aligned}
\beta_{k_1,p_1} &= \beta_{k_2,p_2}\gamma_{k_4}\beta_{k_1,p_1}\varepsilon_{22}\alpha_{k_3,p_3} = \\
&= \beta_{k_2k_4k_1k_5k_3,p_2k_4k_1k_5k_3}.
\end{aligned}$$

As in the previous case we obtain that case (5_{14}) does not hold.

(6_{14}) If $\varepsilon_{11} = \beta_{k_2,p_2}$, $\varepsilon_{12} \in \{\beta_{k_3,p_3}, \gamma_{k_3}, \delta_{k_3}\}$, and $\varepsilon_{22} \in \{\alpha_{k_5,p_5}, \beta_{k_5,p_5}, \gamma_{k_5}, \delta_{k_5}\}$, then by Theorems 1, 3, and 4 and by the similar calculations as in case (4_{14}) we get that

$$\begin{aligned}\beta_{k_1,p_1} &= \beta_{k_2,p_2}\gamma_{k_4}\beta_{k_1,p_1}\varepsilon_{22}\varepsilon_{12} = \\ &= \beta_{k_2k_4k_1k_5k_3,p_2k_4k_1k_5k_3}.\end{aligned}$$

As in case (4_{14}) we obtain that case (6_{14}) does not hold.

(15) Suppose to the contrary that there exist positive integers n_1 and $k_1 \geqslant 2$, and $p_1 \in \{1, \ldots, k_1 - 1\}$ such that the relation $\beta_{k_1,p_1} \mathscr{J} \delta_{n_1}$ holds in $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$. By Theorem 3(8) of [9], $\delta_{n_1} \mathscr{D} \gamma_{n_1}$ in $\boldsymbol{End}^*(\boldsymbol{B}_\omega^{\mathscr{F}})$. This implies that $\delta_{n_1} \mathscr{D} \gamma_{n_1}$ in $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$. Since $\mathscr{D} \subseteq \mathscr{J}$ and $\mathscr{D} \circ \mathscr{J} \subseteq \mathscr{J}$, we obtain that $\beta_{k_1,p_1} \mathscr{J} \gamma_{n_1}$ in $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$, which contradicts statement (14). □

We summarise the descriptions of equivalent classes of Green's relations of the semigroup $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$ in the following theorem.

Theorem 5. *Let $\mathscr{F} = \{[0), [1)\}$. Then the following statements hold.*

1. $\boldsymbol{L}_{\alpha_{k,p}} = \boldsymbol{R}_{\alpha_{k,p}} = \boldsymbol{H}_{\alpha_{k,p}} = \boldsymbol{D}_{\alpha_{k,p}} = \boldsymbol{J}_{\alpha_{k,p}} = \{\alpha_{k,p}\}$ *in* $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$ *for any positive integer k and $p \in \{0, \ldots, k-1\}$.*
2. $\boldsymbol{L}_{\beta_{k,p}} = \{\beta_{k,p}\}$ *and* $\boldsymbol{L}_{\gamma_m} = \boldsymbol{L}_{\delta_m} = \{\gamma_m, \delta_m\}$ *in* $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$ *for any positive integers $k \geqslant 2$, m, and $p \in \{1, \ldots, k-1\}$.*
3. $\boldsymbol{R}_{\beta_{k,p}} = \{\beta_{k,p}\}$, $\boldsymbol{R}_{\gamma_n} = \{\gamma_n\}$, *and* $\boldsymbol{R}_{\delta_n} = \{\delta_n\}$ *in* $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$ *for any positive integers n, $k \geqslant 2$, and $p \in \{1, \ldots, k-1\}$.*
4. *The relation $\mathscr{H}$ on* $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$ *coincides with the equality relation.*
5. $\boldsymbol{D}_{\beta_{k,p}} = \{\beta_{k,p}\}$ *and* $\boldsymbol{D}_{\gamma_m} = \boldsymbol{D}_{\delta_m} = \{\gamma_m, \delta_m\}$ *in* $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$ *for any positive integers $k \geqslant 2$, m, and $p \in \{1, \ldots, k-1\}$.*
6. $\boldsymbol{J}_{\beta_{k,p}} = \{\beta_{k,p}\}$ *and* $\boldsymbol{J}_{\gamma_m} = \boldsymbol{J}_{\delta_m} = \{\gamma_m, \delta_m\}$ *in* $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$ *for any positive integers $k \geqslant 2$, m, and $p \in \{1, \ldots, k-1\}$, and hence $\mathscr{D} = \mathscr{J}$ in* $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$.

Proof. Statement (1) follows from Proposition 2(1) and the inclusion relations between Green's relations.

(2) The equality $\boldsymbol{L}_{\beta_{k,p}} = \{\beta_{k,p}\}$ follows from statements (2)–(4) of Proposition 2 and statement (1). Also, statements (2), (3) and (5)–(7) of Proposition 2 and statement (1) imply that $\boldsymbol{L}_{\gamma_m} = \boldsymbol{L}_{\delta_m} = \{\gamma_m, \delta_m\}$ in $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$ for any positive integers $k \geqslant 2$, m, and $p \in \{1, \ldots, k-1\}$.

(3) The equality $\boldsymbol{R}_{\beta_{k,p}} = \{\beta_{k,p}\}$ follows from statements (8)–(10) of Proposition 2 and statement (1). Also, statements (11) and (12) of Proposition 2 and statement (1) imply the equality $\boldsymbol{R}_{\gamma_n} = \{\gamma_n\}$. The equality $\boldsymbol{R}_{\delta_n} = \{\delta_n\}$ follows from statements (12) and (13) of Proposition 2 and statement (1).

(4) follows from statements (2) and (3) and the definition of $\mathscr{H}$.

(5) follows from statements (2) and (3) and the definition of $\mathscr{D}$.

(6) The equality $\boldsymbol{J}_{\beta_{k,p}} = \{\beta_{k,p}\}$ follows from statements (14) and (15) of Proposition 2 and statement (1).

Next we shall prove that $\boldsymbol{J}_{\gamma_m} = \boldsymbol{J}_{\delta_m} = \{\gamma_m, \delta_m\}$ in $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$ for any positive integer m, which with the above statements imply that $\mathscr{D} = \mathscr{J}$ in $\boldsymbol{End}(\boldsymbol{B}_\omega^{\mathscr{F}})$. We observe that by statement (5) it is complete to show that $\gamma_n \mathscr{J} \gamma_m$ in $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$, $n,m \in \mathbb{N}$, if and only if $n=m$.

Suppose to the contrary that $\gamma_n \mathscr{J} \gamma_m$ in $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ for distinct $n,m \in \mathbb{N}$. Without loss of generality we may assume that $n<m$. Then by the definition of the relation $\mathscr{J}$ there exist $\varepsilon_{11},\varepsilon_{12},\varepsilon_{21},\varepsilon_{22} \in \boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})^1$ such that $\gamma_n=\varepsilon_{11}\gamma_m\varepsilon_{12}$ and $\gamma_m=\varepsilon_{21}\gamma_n\varepsilon_{22}$. By Theorem 3, $\varepsilon_{11},\varepsilon_{21} \notin \langle\delta\rangle$. By Theorem 4 we get that $\varepsilon_{11} \notin \langle\alpha\rangle \cup \langle\beta\rangle$, and hence $\varepsilon_{11}=\gamma_k$ for some positive integer k. Then by Theorems 3 and 4 the equality $\gamma_n=\varepsilon_{11}\gamma_m\varepsilon_{12}=\gamma_k\gamma_m\varepsilon_{12}$ implies that $n \geqslant m$, which contradicts our assumption. □

Remark 2. Theorem 2 of [8], Theorem 3 of [9] and Theorem 5 imply that for an ω-closed family $\mathscr{F}$ which consists of two nonempty inductive nonempty subsets of ω the following statements holds:

(1) Green's relations of elements α_{k_1,p_1} and β_{k_2,p_2} in the semigroup $\boldsymbol{End}_*^1(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of all injective monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ coincide with the corresponding their Green's relations in the semigroup $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of all monoid endomorphisms of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$.

(2) Green's relations of elements γ_{n_1} and δ_{n_2} in the semigroup $\boldsymbol{End}_0^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of all non-injective monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ coincide with the corresponding their Green's relations in the semigroup $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of all monoid endomorphisms of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$.

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ПРО НАПІВГРУПУ УСІХ МОНОЇДАЛЬНИХ ЕНДОМОРФІЗМІВ НАПІВГРУПИ $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ З ДВОЕЛЕМЕНТНОЮ СІМ'ЄЮ $\mathscr{F}$ ІНДУКТИВНИХ НЕПОРОЖНІХ ПІДМНОЖИН У ω

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Вивчено структуру напівгрупи $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ усіх моноїдальних ендоморфізмів напівгрупи $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, де ω-замкнена сім'я $\mathscr{F}$ складається з двох непорожніх індуктивних підмножин у ω. Описано елементи напівгрупи $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$, напівгрупову операцію та відношення Ґріна на напівгрупі $\boldsymbol{End}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$.

Ключові слова: біциклічний моноїд, інверсна напівгрупа, біциклічне розширення, ендоморфізм, ідеал, відношення Ґріна.